SOLUTION OF HEAT CONDUCTION PROBLEMS FOR A BODY OF REVOLUTION MADE OF INHOMOGENEOUS MATERIAL

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Solutions of the heat conduction equation are obtained which permit methods of solving plane boundary problems for analytic functions to be applied when the boundary conditions on a surface of revolution are satisfied.

Let us consider a body V generated by rotation of a plane region D about an axis z. The material of the body is inhomogeneous and anisotropic, its thermal conductivities in the directions of the axes ρ, z, θ being, respectively, λ_1, λ_2, λ_3, which are differentiable functions of ρ:

$$\lambda_j = \lambda_j(\rho), \quad j = 1,\ 2,\ 3. \tag{1}$$

For steady conditions in an inhomogeneous, anisotropic medium, taking (1) into account, the heat conduction equation may be written in the form

$$N \equiv \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\lambda_1\rho\frac{\partial T}{\partial\rho}\right) + \frac{\partial}{\partial z}\left(\lambda_2\frac{\partial T}{\partial z}\right) + \frac{1}{\rho}\frac{\partial}{\partial\theta}\left(\frac{\lambda_3}{\rho}\frac{\partial T}{\partial\theta}\right) \equiv \lambda_1\left[\frac{\partial^2}{\partial\rho^2} + \frac{\partial}{\partial\rho}(\ln\rho\lambda_1)\frac{\partial}{\partial\rho} + \frac{\lambda_2}{\lambda_1}\frac{\partial^2}{\partial z^2} + \frac{\lambda_3}{\rho^2\lambda_1}\frac{\partial^2}{\partial\theta^2}\right]T = 0. \tag{2}$$

We shall look for temperature T in the form

$$T = u(\rho)\Phi, \tag{3}$$

where $u(\rho)$ is a function of ρ, and Φ is a function of the variables ρ, z, θ. We designate

$$u = u(\rho), \quad \omega = \omega(\rho), \quad \xi = \omega + iz, \quad \bar{\xi} = \omega - iz. \tag{4}$$

We note that

$$\frac{\partial\Phi}{\partial\rho} = \omega'\left(\frac{\partial\Phi}{\partial\xi} + \frac{\partial\Phi}{\partial\bar{\xi}}\right), \quad \frac{\partial^2\Phi}{\partial\rho^2} = \frac{\omega''}{\omega'}\frac{\partial\Phi}{\partial\rho} + (\omega')^2\left(\frac{\partial^2\Phi}{\partial\xi^2} + \frac{\partial^2\Phi}{\partial\bar{\xi}^2} + 2\frac{\partial^2\Phi}{\partial\xi\partial\bar{\xi}}\right),$$

$$\frac{\partial^2\Phi}{\partial z^2} = 2\frac{\partial^2\Phi}{\partial\xi\partial\bar{\xi}} - \frac{\partial^2\Phi}{\partial\xi^2} - \frac{\partial^2\Phi}{\partial\bar{\xi}^2},$$

$$4\frac{\partial^2\Phi}{\partial\xi\partial\bar{\xi}} = \frac{\partial^2\Phi}{\partial\omega^2} + \frac{\partial^2\Phi}{\partial z^2}. \tag{5}$$

Substituting (3) into (2), and taking account of (5) and (4), we transform (2) into the form

$$N \equiv u\lambda_1\left\{2\left[(\omega')^2 + \frac{\lambda_2}{\lambda_1}\right]\frac{\partial^2}{\partial\xi\partial\bar{\xi}} + \left[(\omega')^2 - \frac{\lambda_2}{\lambda_1}\right]\left(\frac{\partial^2}{\partial\xi^2} + \frac{\partial^2}{\partial\bar{\xi}^2}\right) + \frac{\partial}{\partial\rho}(\ln\rho\lambda_1\omega' u^2)\frac{\partial}{\partial\rho} + g(u) + \frac{\lambda_3}{\rho^2\lambda_1}\frac{\partial^2}{\partial\theta^2}\right\}\Phi = 0, \tag{6}$$

where

$$g(u) = \frac{u''}{u} + \frac{u'}{u}\frac{\partial}{\partial\rho}\ln\rho\lambda_1. \tag{7}$$

We put

$$\lambda_2 = (\omega')^2\lambda_1, \quad \lambda_1\rho\omega' u^2 = 1,$$

and then

$$g(u) = u''/u - 2(u/u)^2 - \omega'' u'/\omega' u. \tag{8}$$

Taking account of (5), we rewrite (6) in the form

$$N \equiv u\lambda_2\left[\frac{\partial^2}{\partial\omega^2} + \frac{\partial^2}{\partial z^2} + \frac{\lambda_1}{\lambda_2}g(u) + \frac{\lambda_3}{\rho^2\lambda_2}\frac{\partial^2}{\partial\theta^2}\right]\Phi = 0. \tag{9}$$

We shall examine the solution of (9) for certain cases.

1. The temperature is independent of θ, a

$$\lambda_2(\rho) = c^2\lambda_1(\rho); \quad c = \text{const}, \quad \omega(\rho) = c\rho. \tag{10}$$

We require that

$$g(u) = 0. \tag{11}$$

For conditions (10) the general solution of (11) and (8) is

$$u(\rho) = \frac{1}{\sqrt{cc_0}}(\rho + a)^{-1}, \tag{12}$$

where a, c_0 are arbitrary positive constants. For $\lambda_1(\rho)$, taking account of (10), (12), and (8), we obtain the formula

$$\lambda_1(\rho) = \frac{c_0}{\rho}(\rho + a)^2. \tag{13}$$

Since the temperature (3) does not depend on θ, taking (10) and (11) into account, we obtain the solution of (9) in the form

$$\Phi = \varphi(\xi) + \overline{\varphi(\xi)}, \quad \xi = c\rho + iz.$$

Then for temperature (3) we obtain the expression

$$T = u(\rho)[\varphi(\xi) + \overline{\varphi(\xi)}]. \quad (14)$$

Thus, we satisfy (2) for conditions (10)–(14), and it is clear that the solution of the boundary heat conduction problem with a boundary condition of the first kind on the surface of the body V may be accomplished by known methods of the Dirichlet plane boundary problem, this being so for a very wide class of regions D.

It may be shown that in some range of variation of ρ, depending on the parameter a, i.e., when

$$0 < \rho_1 \leqslant \rho \leqslant \rho_2, \quad (15)$$

function (13) takes values close to constant. Therefore in the range (15), solution (14) is applicable for investigation of the temperature of bodies of inhomogeneous material. Indeed, function (13) has a minimum at point $\rho = a$, i.e., $\lambda_1(\rho) > \lambda_1(a)$ when $\rho < a$, $\rho > a$. We shall require that

$$\lambda_1(\rho) = \lambda_1[1 + \Pi(\rho)],$$

$$|\Pi(\rho)| \leqslant \delta \text{ when } \rho_1 \leqslant \rho \leqslant \rho_2, \quad \rho_1 < a < \rho_2, \quad (16)$$

where δ is a sufficiently small positive quantity, and λ is a constant. To satisfy (16) it is sufficient to determine constants c_0, ρ_1, ρ_2 such that

$$\lambda_1(a) = \lambda(1 - \delta), \quad \lambda_1(\rho_1) = \lambda_1(\rho_2) = \lambda(1 + \delta). \quad (17)$$

We put

$$c_0 = \lambda\rho_1(1 + \delta)(\rho_1 + a)^{-2} \quad (18)$$

and substituting in (13), taking account of (18) and (17), we shall satisfy (17) when

$$\rho_1 = \frac{a}{1 - \delta}(\sqrt{1+\delta} - \sqrt{2\delta})^2,$$

$$\rho_2 = \rho_1\left(\frac{\sqrt{1+\delta} + \sqrt{2\delta}}{\sqrt{1+\delta} - \sqrt{2\delta}}\right)^2. \quad (19)$$

We shall derive values of ρ_1 and ρ_2 when $\delta = 0.05$ and $\delta = 0.1$, i.e., for the cases when the maximum deviations of $\lambda_1(\rho)$ from λ = const in the range (32) are, respectively, 5 and 10%:

$$\delta = 0.05, \quad \rho_1 = 0.5285a, \quad \rho_2 = 3.58\rho_1;$$

$$\delta = 0.1, \quad \rho_1 = 0.4024a, \quad \rho_2 = 6.18\rho_1. \quad (20)$$

Since real bodies are not in most cases absolutely inhomogeneous, (14) may be used even to investigate the temperature of bodies which are usually considered homogeneous. Of course, for calculations with actual bodies, standards for δ must be established which satisfy the limits (19) and (15), where (14) has been applied. In particular, if it is allowable to put $\delta = 0.05$, then (14) may be used to investigate homogeneous bodies having a region D within the limits of (15) and (20), where a is an arbitrary positive constant, i.e., $a > 0$.

2. The temperature does not depend on θ and a:

$$\lambda_1(\rho) = \lambda = \text{const}. \quad (21)$$

We are required to satisfy (11). Taking account of (21) and satisfying (11) and (8), we obtain

$$\lambda_2(\rho) = c_1/\rho^2(\ln a\rho)^4, \quad \omega(\rho) = -\sqrt{c_1}/\sqrt{\lambda}\ln a\rho,$$

$$u(\rho) = (\lambda c_1)^{-1/4}\ln a\rho, \quad (22)$$

where a and c_1 are arbitrary positive constants. Here also we shall have (14) and take account of (4) and (22) for the temperature.

3. We have

$$\omega'(\rho) > 0, \quad \omega(\rho) \neq -a_1. \quad (23)$$

The general solution of (11) and (8) may be written in the form

$$u(\rho) = \frac{1}{\sqrt{c_2}}[\omega(\rho) + a_1]^{-1}, \quad (24)$$

and according to (8) we obtain

$$\lambda_1(\rho) = c_2\frac{[\omega(\rho) + a_1]^2}{\rho\omega'(\rho)},$$

$$\lambda_2(\rho) = \frac{c_2}{\rho}\omega'(\rho)[\omega(\rho) + a_1]^2, \quad (25)$$

where c_2 and a_1 are arbitrary constants. When conditions (23)–(25) are fulfilled, the temperature (14) satisfies (9).

4. The temperature depends on θ on a:

$$\lambda_2(\rho) = \lambda_1(\rho), \quad \lambda_3(\rho) = \frac{\lambda_1(\rho)}{4\mu^2}, \quad \mu - \text{constant}. \quad (26)$$

Here $\omega(\rho) = \rho$, and (9) takes the form

$$N \equiv u\lambda_1\Lambda\Phi = 0,$$

$$\Lambda = \frac{\partial^2}{\partial\rho^2} + \frac{\partial^2}{\partial z^2} + g(u) + \frac{1}{4\mu^2\rho^2}\frac{\partial^2}{\partial\theta^2}. \quad (27)$$

We shall seek the function Φ in the form

$$\Phi = \operatorname{Re}\sum_{s=0}^{n}\gamma_s(\theta)\Phi_s,$$

$$\Phi_s = \sum_{k=0}^{s} a_{k,s}\rho^{-k}\int_k \varphi_s(\xi)\,d\xi, \quad \xi = \rho + iz, \quad (28)$$

where the $a_{k,s}$ are constants, and the function $\gamma_s(\theta)$ satisfies the equation

$$\gamma_s''(\theta) + \mu_s^2\gamma_s(\theta) = 0, \quad \mu_s = \mu(2s + 1). \quad (29)$$

Substituting (28) into (27) and taking (29) into account, we have

$$\Lambda[\gamma_s(\theta)\Phi_s] = \gamma_s(\theta)\left[\frac{\partial^2}{\partial\rho^2} + \frac{\partial^2}{\partial z^2} + g(u) - \frac{\mu_s^2}{\rho^2}\right]\Phi_s =$$

$$\equiv \gamma_s(0)\sum_{k=0}^{s} a_{k,s}\rho^{-k-2}\left\{\left[k^2 + k + \rho^2 g(u) - \frac{(2s+1)^2}{4}\right]\times\right.$$

$$\left.\times\int_k \varphi_s(\xi)\,d\xi - 2k\rho\int_{k-1}\varphi_s(\xi)\,d\xi\right\} \equiv$$

$$= \gamma_s(\theta)\sum_{k=0}^{s}\rho^{-k-2}\left\{a_{k,s}\left[k^2 + k + \rho^2 g(u) -\right.\right.$$

$$\left.\left. - \frac{1}{4} - s^2 - s\right] - 2(k+1)a_{k+1,s}\right\}\int_k \varphi_s(\xi)\,d\xi = 0. \quad (30)$$

If we put

$$a_{0,s} = 1, \quad a_{k+1,s} = \frac{a_{k,s}}{2(k+1)}(k^2 + k - s^2 - s),$$

$$k = 0,\ 1, \ldots,\ s-1 \quad (31)$$

and require that

$$\rho^2 g(u) - 1/4 \equiv \rho^2[u''/u - 2(u'/u)^2] - 1/4 = 0, \quad (32)$$

then (30) will be satisfied. The general solution of (32) is

$$u(\rho) = c_1(1/\sqrt{\rho} + 1/(c - \ln\rho)), \quad (33)$$

where c_1 and c are arbitrary constants. From (8), taking (33) into account, we obtain

$$\lambda_1(\rho) = \frac{1}{\rho c_1^2}\left(\frac{1}{\sqrt{\rho}} + \frac{1}{c - \ln\rho}\right)^{-2}. \quad (34)$$

If $c = \infty$,

$$\lambda_1(\rho) = 1/c_1^2 = \text{const}. \quad (35)$$

Thus, under conditions (33), (34), (26), (29), and (31), function (28) is a solution of (27). Let the temperature on the surface of the body V be given as

$$T = u(\rho)\sum_{s=0}^{n}\gamma_s(\theta)\,p_s, \quad (36)$$

where p_s are functions of the arc of the boundary of region D. Taking into account (3) and (28), we shall satisfy boundary conditions (36), after solving the plane boundary problems

$$\mathrm{Re}\Phi_s = p_s, \quad s = 0,\ 1,\ 2,\ \ldots,\ n$$

with respect to the analytic functions $\varphi_s(\xi)$.

NOTATION

$u' = \frac{\partial u}{\partial\rho}$, $u'' = \frac{\partial^2 u}{\partial\rho^2}$, Re F—real part of F; $\varphi(\xi)$, $\varphi_s(\xi)$—analytic functions of the complex variable (4); $\int_k \varphi\,d\xi$—the operation of integrating k times with respect to ξ, i.e., $\int_0 \varphi\,d\xi = \varphi$; $\int_1 \varphi\,d\xi = \int\varphi\,d\xi$; $\int_2 \varphi\,d\xi = \int d\xi\int\varphi\,d\xi$ etc.

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